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Inverse Radiation Problem in Two-Dimensional Rectangular Media

H. Y. Li*
Hua Fan University, Shihtin, Taipei, Taiwan 22305, Republic of China

An inverse radiation analysis is presented for estimating the unknown source term in a two-dimensional absorbing, emitting, scattering rectangular medium of known optical properties from the knowledge of the radiation intensities exiting the boundaries. The inverse problem is solved by using a least-squares method that minimizes the error between the exit radiation intensities calculated and the experimental measurements. The effects of the measurement errors, anisotropic scattering, single-scattering albedo, and optical thickness on the accuracy of the inverse analysis are investigated. Although the source term is a function of the space variables, the inverse algorithm presented requires measurements of the outgoing radiation intensities only at the center of each boundary. The study shows that the source term can be estimated accurately, even with noisy data.

Nomenclature

A = area, Eq. (3c)
a_n = expansion coefficients for the phase function
B = area, Eq. (3d)
$\boldsymbol{b}$ = $(b_{00}, b_{10}, \ldots, b_{MN})^T$
b_{qr} = expansion coefficients for the source term
$\boldsymbol{d}$ = direction of descent
$f_q(y)$ = basis function
$g_r(z)$ = basis function
H = optical thickness of the rectangle in the z direction
I = radiation intensity
J = objective function
M = upper limit of the outer summation, Eq. (1h)
N = upper limit of the inner summation, Eq. (1h)
N^* = order of the phase function
$\bar{n}$ = refractive index
P_n = Legendre polynomials
p = scattering phase function
S = source term, Eq. (1g)
S^* = function, Eq. (3b)
T = temperature
V = volume, Eq. (3e)
W = optical thickness of the rectangle in the y direction
w = quadrature weight
Y = measured exit radiation intensity
y, z = optical coordinates
∇I = sensitivity coefficient vector
∇J = gradient of the objective function
$\boldsymbol{\beta}$ = step size
γ = conjugate coefficient
ζ = random variable
θ = polar angle
ξ, η, μ = direction cosines of $\boldsymbol{\Omega}$
σ = standard deviation
$\bar{\sigma}$ = Stefan–Boltzmann constant
$\boldsymbol{\tau}$ = position vector
ϕ = azimuthal angle
Ω = solid angle
$\boldsymbol{\Omega}$ = direction of the radiation intensity
ω = single-scattering albedo

Subscripts

i = position index in the y direction
j = position index in the z direction
m = direction used in the discrete ordinates equations
w = wall value
1 = position at $(0, H/2)$
2 = position at $(W, H/2)$
3 = position at $(W/2, 0)$
4 = position at $(W/2, H)$

Subscripts

k = kth iteration
T = transpose

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*Associate Professor, Department of Mechanical Engineering.

Introduction

INVERSE radiation problems are concerned with the determination of the radiative properties of media using various types of radiation measurements. The problems have received much attention in the past, and there is a good deal of research surrounding this topic.[1–7] Reviews of the inverse techniques may be found in a series of papers by McCormick.[8–10] Inverse problems that deal with the prediction of the temperature profile in a medium from radiation measurements have also been reported by many researchers.[11–15] Yi et al.[11] determined the unknown source term that requires the measurements to be made inside the medium. From the experimental point of view, it is desirable to avoid detectors within the medium. Li and Ozisik,[12] Siewert,[13,14] and Li[15] have reconstructed the temperature distributions in plane–parallel, spherical, and cylindrical media by the inverse analysis that requires only the data of the radiation intensities exiting the boundaries. Despite the relatively large interest expressed in inverse radiation problems, all of the work has considered one-dimensional systems.

In this paper, we deal with the inverse problem of estimating the source term in a two-dimensional absorbing, emitting, scattering rectangular medium from measured radiation intensities exiting the boundaries. The inverse problem is solved by using a least-squares method that minimizes the error between the exit radiation intensities calculated and the experimental data. The analysis involves the direct problem, sensitivity problem, and the gradient equation. The effects of the measurement errors, anisotropic scattering, single-scattering albedo, and optical thickness on the accuracy of the inverse analysis will be examined.

Analysis

Direct Problem

Consider a two-dimensional absorbing, emitting, scattering, and gray rectangular medium confined to the domain $0 \leq y \leq W$, $0 \leq z \leq H$, as illustrated in Fig. 1. The boundary surfaces are considered to be transparent, and there is no external incident radiation. The equation of radiative transfer can be written as follows[16,17]:

$$\eta \frac{\partial I(\boldsymbol{\tau}, \boldsymbol{\Omega})}{\partial y} + \mu \frac{\partial I(\boldsymbol{\tau}, \boldsymbol{\Omega})}{\partial z} + I(\boldsymbol{\tau}, \boldsymbol{\Omega}) = S(\boldsymbol{\tau}) + \frac{\omega}{4\pi} \int_{\Omega'=4\pi} p(\boldsymbol{\Omega}', \boldsymbol{\Omega}) I(\boldsymbol{\tau}, \boldsymbol{\Omega}')\, \mathrm{d}\Omega' \tag{1a}$$

with the boundary conditions

$$I(\boldsymbol{\tau}_w, \boldsymbol{\Omega}) = 0, \qquad y = 0, \qquad \eta > 0 \tag{1b}$$

$$I(\boldsymbol{\tau}_w, \boldsymbol{\Omega}) = 0, \qquad y = W, \qquad \eta < 0 \tag{1c}$$

$$I(\boldsymbol{\tau}_w, \boldsymbol{\Omega}) = 0, \qquad z = 0, \qquad \mu > 0 \tag{1d}$$

$$I(\boldsymbol{\tau}_w, \boldsymbol{\Omega}) = 0, \qquad z = H, \qquad \mu < 0 \tag{1e}$$

where $I(\boldsymbol{\tau}, \boldsymbol{\Omega})$ is the radiation intensity at the location $\boldsymbol{\tau}$ in the direction $\boldsymbol{\Omega}$. $\boldsymbol{\Omega}$ is defined by the direction cosines ξ, η, and μ, given by

$$\xi = \sin\theta \cos\phi, \qquad \eta = \sin\theta \sin\phi, \qquad \mu = \cos\theta \tag{1f}$$

The source term $S(\boldsymbol{\tau})$ is related to the temperature $T(\boldsymbol{\tau})$ in the medium by

$$S(\boldsymbol{\tau}) = (1 - \omega)[\bar{n}^2 \bar{\sigma} T^4(\boldsymbol{\tau})/\pi] \tag{1g}$$

it is represented as

$$S(\boldsymbol{\tau}) = \sum_{q=0}^{M} \sum_{r=0}^{N} b_{qr} f_q(y) g_r(z) \tag{1h}$$

where $f_q(y)$ and $g_r(z)$ are basis functions. The scattering phase function $p(\boldsymbol{\Omega}', \boldsymbol{\Omega})$ is expressed in terms of the Legendre polynomials as

$$p(\boldsymbol{\Omega}', \boldsymbol{\Omega}) = \sum_{n=0}^{N^*} a_n P_n(\xi'\xi + \eta'\eta + \mu'\mu) \tag{1i}$$

where a_n are the coefficients of the expansion, and N^* is the order of the phase function. The direct problem of concern here is to find the radiation intensity for the known source term, phase function, radiation properties, and boundary conditions. We apply the S_N or the discrete ordinates method[17,18] to solve the direct problem. In this approach, the solid angle 4π is discretized into a finite number of directions. The equation of radiative transfer is evaluated at each of the discrete directions, and the integral term is replaced by a weighted sum, which leads to the discrete ordinates equations

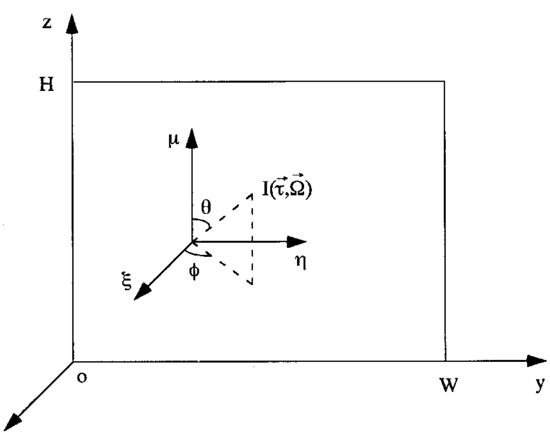

Fig. 1 Schematic of the physical system and coordinates.

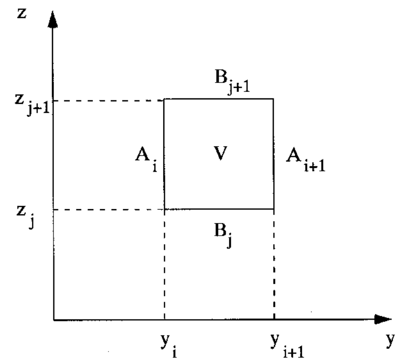

Fig. 2 Mesh cell in two-dimensional rectangular geometry.

$$\eta_m \frac{\partial I_m}{\partial y} + \mu_m \frac{\partial I_m}{\partial z} + I_m = S(\boldsymbol{\tau}) + \frac{\omega}{4\pi} \sum_{m'} w_{m'} p_{m'm} I_{m'} \tag{2a}$$

with the boundary conditions

$$I_m = 0, \qquad \eta_m > 0, \qquad y = 0 \tag{2b}$$

$$I_m = 0, \qquad \eta_m < 0, \qquad y = W \tag{2c}$$

$$I_m = 0, \qquad \mu_m > 0, \qquad z = 0 \tag{2d}$$

$$I_m = 0, \qquad \mu_m < 0, \qquad z = H \tag{2e}$$

where subscripts m and m' represent the discrete directions, w_m are the quadrature weights, and the phase function is given by

$$p_{m'm} = \sum_{n=0}^{N^*} a_n P_n(\xi_{m'}\xi_m + \eta_{m'}\eta_m + \mu_{m'}\mu_m) \tag{2f}$$

A control volume form of the discrete ordinates equations can be obtained by integrating Eq. (2a) over the cell from $y = y_i$ to y_{i+1} and $z = z_j$ to z_{j+1}, as shown in Fig. 2, from which we can show that

$$\eta_m(A_{i+1} I_{m,i+1} - A_i I_{m,i}) + \mu_m(B_{j+1} I_{m,j+1} - B_j I_{m,j}) + V I_m = V S_m^* \tag{3a}$$

where

$$S_m^* = S + \frac{\omega}{4\pi} \sum_{m'} w_{m'} p_{m'm} I_{m'} \tag{3b}$$

$$A_{i+1} = A_i = z_{j+1} - z_j \tag{3c}$$

$$B_{j+1} = B_j = y_{i+1} - y_i \tag{3d}$$

$$V = (y_{i+1} - y_i)(z_{j+1} - z_j) \tag{3e}$$

Here, I_m and S_m^* are evaluated at the center of cell V. If the intensity is assumed to vary linearly with all of the independent variables, then the intensity at the cell center I_m, the intensities $I_{m,i+1}$, $I_{m,i}$ at the cell boundaries $i + 1$ and i, and the intensities $I_{m,j+1}$, $I_{m,j}$ at the cell boundaries $j + 1$ and j are related by

$$I_{m,i} + I_{m,i+1} = I_{m,j} + I_{m,j+1} = 2I_m \tag{3f}$$

Table 1 Quadrature points and weights for the S_6 scheme

m	ξ_m	η_m	μ_m	w_m
1	0.224556	−0.224556	−0.948235	$\pi/6$
2	0.224556	0.224556	−0.948235	$\pi/6$
3	0.224556	−0.689048	−0.689048	$\pi/6$
4	0.689048	−0.224556	−0.689048	$\pi/6$
5	0.689048	0.224556	−0.689048	$\pi/6$
6	0.224556	0.689048	−0.689048	$\pi/6$
7	0.224556	−0.948235	−0.224556	$\pi/6$
8	0.689048	−0.689048	−0.224556	$\pi/6$
9	0.948235	−0.224556	−0.224556	$\pi/6$
10	0.948235	0.224556	−0.224556	$\pi/6$
11	0.689048	0.689048	−0.224556	$\pi/6$
12	0.224556	0.948235	−0.224556	$\pi/6$
13	0.224556	−0.948235	0.224556	$\pi/6$
14	0.689048	−0.689048	0.224556	$\pi/6$
15	0.948235	−0.224556	0.224556	$\pi/6$
16	0.948235	0.224556	0.224556	$\pi/6$
17	0.689048	0.689048	0.224556	$\pi/6$
18	0.224556	0.948235	0.224556	$\pi/6$
19	0.224556	−0.689048	0.689048	$\pi/6$
20	0.689048	−0.224556	0.689048	$\pi/6$
21	0.689048	0.224556	0.689048	$\pi/6$
22	0.224556	0.689048	0.689048	$\pi/6$
23	0.224556	−0.224556	0.948235	$\pi/6$
24	0.224556	0.224556	0.948235	$\pi/6$

which is called the diamond difference scheme. If the diamond difference approximation leads to negative intensities, an upwind difference scheme[19] is adopted where positive intensities are guaranteed. The moment-matching technique suggested by Carlson and Lathrop[18] is applied to calculate the quadrature points and weights, and they are listed in Table 1 for the S_6 scheme. Equations (3) and (2b)–(2f) are solved using the procedure described in Ref. 17 to calculate the radiation intensity.

Inverse Problem

For the inverse problem, the source term is regarded as unknown, but other quantities in Eqs. (1) are known. In addition, measured radiation intensities exiting the boundaries are considered available. In the inverse analysis, we estimate the source term by utilizing the measured data. The problem of estimating the unknown source term is solved by minimizing the objective function that is the square deviation between the exit radiation intensities calculated and the experimental measurements

$$J(\boldsymbol{b}) = \int_{\eta<0} [I_1(\boldsymbol{\Omega};\boldsymbol{b}) - Y_1(\boldsymbol{\Omega})]^2\,\mathrm{d}\Omega + \int_{\eta>0} [I_2(\boldsymbol{\Omega};\boldsymbol{b}) - Y_2(\boldsymbol{\Omega})]^2\,\mathrm{d}\Omega + \int_{\mu<0} [I_3(\boldsymbol{\Omega};\boldsymbol{b}) - Y_3(\boldsymbol{\Omega})]^2\,\mathrm{d}\Omega + \int_{\mu>0} [I_4(\boldsymbol{\Omega};\boldsymbol{b}) - Y_4(\boldsymbol{\Omega})]^2\,\mathrm{d}\Omega \tag{4}$$

where $\boldsymbol{\tau}_1 = (0, H/2)$, $\boldsymbol{\tau}_2 = (W, H/2)$, $\boldsymbol{\tau}_3 = (W/2, 0)$, $\boldsymbol{\tau}_4 = (W/2, H)$; $Y_1(\boldsymbol{\Omega})$, $Y_2(\boldsymbol{\Omega})$, $Y_3(\boldsymbol{\Omega})$, and $Y_4(\boldsymbol{\Omega})$ are the measured exit radiation intensities at $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$, and $\boldsymbol{\tau}_4$, respectively; and $I_1(\boldsymbol{\Omega};\boldsymbol{b})$, $I_2(\boldsymbol{\Omega};\boldsymbol{b})$, $I_3(\boldsymbol{\Omega};\boldsymbol{b})$, and $I_4(\boldsymbol{\Omega};\boldsymbol{b})$ are the estimated exit intensities at $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$, and $\boldsymbol{\tau}_4$, respectively, for an estimated vector $\boldsymbol{b} = (b_{00}, b_{10}, \ldots, b_{MN})^T$.

Conjugate Gradient Method of Minimization

The minimization of the objective function with respect to the desired vector is the most important procedure in solving the inverse problem. In this paper, we use the conjugate gradient method[20] to determine the unknown source term $s(\boldsymbol{\tau})$. Iterations are built in the following manner:

$$\boldsymbol{b}^{k+1} = \boldsymbol{b}^k - \boldsymbol{\beta}^k \boldsymbol{d}^k \tag{5}$$

where β^k is the step size, $\boldsymbol{d}^k$ is the direction of descent given by

$$\boldsymbol{d}^k = \nabla J^T(\boldsymbol{b}^k) + \gamma^k \boldsymbol{d}^{k-1} \tag{6}$$

and the conjugate coefficient γ^k is determined from

$$\gamma^k = \frac{\nabla J(\boldsymbol{b}^k)\nabla J^T(\boldsymbol{b}^k)}{\nabla J(\boldsymbol{b}^{k-1})\nabla J^T(\boldsymbol{b}^{k-1})} \quad \text{with} \quad \gamma^0 = 0 \tag{7}$$

Here, the row vector defined by

$$\nabla J = \left(\frac{\partial J}{\partial b_{00}}, \frac{\partial J}{\partial b_{10}}, \ldots, \frac{\partial J}{\partial b_{MN}}\right) \tag{8}$$

is the gradient of the objective function. β^k is determined by minimizing the objective function given by Eq. (4) with respect to β^k, from which it can be shown that

$$\beta^k = \left\{\int_{\eta<0} [I_1(\boldsymbol{\Omega};\boldsymbol{b}^k) - Y_1(\boldsymbol{\Omega})]\nabla I_1(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k\,\mathrm{d}\Omega + \int_{\eta>0} [I_2(\boldsymbol{\Omega};\boldsymbol{b}^k) - Y_2(\boldsymbol{\Omega})]\nabla I_2(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k\,\mathrm{d}\Omega + \int_{\mu<0} [I_3(\boldsymbol{\Omega};\boldsymbol{b}^k) - Y_3(\boldsymbol{\Omega})]\nabla I_3(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k\,\mathrm{d}\Omega + \int_{\mu>0} [I_4(\boldsymbol{\Omega};\boldsymbol{b}^k) - Y_4(\boldsymbol{\Omega})]\nabla I_4(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k\,\mathrm{d}\Omega\right\} \Bigg/ \left\{\int_{\eta<0} [\nabla I_1(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k]^2\,\mathrm{d}\Omega + \int_{\eta>0} [\nabla I_2(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k]^2\,\mathrm{d}\Omega + \int_{\mu<0} [\nabla I_3(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k]^2\,\mathrm{d}\Omega + \int_{\mu>0} [\nabla I_4(\boldsymbol{\Omega};\boldsymbol{b}^k)\boldsymbol{d}^k]^2\,\mathrm{d}\Omega\right\} \tag{9}$$

where the row vector

$$\nabla I = \left(\frac{\partial I}{\partial b_{00}}, \frac{\partial I}{\partial b_{10}}, \ldots, \frac{\partial I}{\partial b_{MN}}\right) \tag{10}$$

is the sensitivity coefficient vector. To perform the iteration given by Eqs. (5)–(10), we need to compute the gradient of the objective function ∇J and the sensitivity coefficient vector ∇I.

Sensitivity Problem

The sensitivity problem is obtained by differentiating the direct problem given by Eqs. (1) with respect to b_{qr}, from which we can show that

$$\eta \frac{\partial}{\partial y}\left[\frac{\partial I(\boldsymbol{\tau},\boldsymbol{\Omega})}{\partial b_{qr}}\right] + \mu \frac{\partial}{\partial z}\left[\frac{\partial I(\boldsymbol{\tau},\boldsymbol{\Omega})}{\partial b_{qr}}\right] + \left[\frac{\partial I(\boldsymbol{\tau},\boldsymbol{\Omega})}{\partial b_{qr}}\right] = f_q(y)g_r(z) + \frac{\omega}{4\pi}\int_{\Omega'=4\pi} p(\boldsymbol{\Omega}',\boldsymbol{\Omega})\frac{\partial I(\boldsymbol{\tau},\boldsymbol{\Omega}')}{\partial b_{qr}}\,\mathrm{d}\Omega' \tag{11a}$$

with the boundary conditions

$$\frac{\partial I(\boldsymbol{\tau}_w, \boldsymbol{\Omega})}{\partial b_{qr}} = 0, \qquad y = 0, \qquad \eta > 0 \tag{11b}$$

$$\frac{\partial I(\boldsymbol{\tau}_w, \boldsymbol{\Omega})}{\partial b_{qr}} = 0, \qquad y = W, \qquad \eta < 0 \tag{11c}$$

$$\frac{\partial I(\boldsymbol{\tau}_w, \boldsymbol{\Omega})}{\partial b_{qr}} = 0, \qquad z = 0, \qquad \mu > 0 \tag{11d}$$

$$\frac{\partial I(\boldsymbol{\tau}_w, \boldsymbol{\Omega})}{\partial b_{qr}} = 0, \qquad z = H, \qquad \mu < 0 \tag{11e}$$

for $q = 0, 1, \ldots, M$, $r = 0, 1, \ldots, N$. The solution procedure for ∇I is the same as that for the direct problem, so it will not be repeated here.

Gradient Equation

To develop expressions for the gradient ∇J, we differentiate J with respect to b_{qr} to obtain

$$\begin{aligned}\frac{\partial J}{\partial b_{qr}} &= 2\int_{\eta<0}[I_1(\boldsymbol{\Omega};\boldsymbol{b}) - Y_1(\boldsymbol{\Omega})]\frac{\partial I_1(\boldsymbol{\Omega};\boldsymbol{b})}{\partial b_{qr}}\,\mathrm{d}\Omega \\ &+ 2\int_{\eta>0}[I_2(\boldsymbol{\Omega};\boldsymbol{b}) - Y_2(\boldsymbol{\Omega})]\frac{\partial I_2(\boldsymbol{\Omega};\boldsymbol{b})}{\partial b_{qr}}\,\mathrm{d}\Omega \\ &+ 2\int_{\mu<0}[I_3(\boldsymbol{\Omega};\boldsymbol{b}) - Y_3(\boldsymbol{\Omega})]\frac{\partial I_3(\boldsymbol{\Omega};\boldsymbol{b})}{\partial b_{qr}}\,\mathrm{d}\Omega \\ &+ 2\int_{\mu>0}[I_4(\boldsymbol{\Omega};\boldsymbol{b}) - Y_4(\boldsymbol{\Omega})]\frac{\partial I_4(\boldsymbol{\Omega};\boldsymbol{b})}{\partial b_{qr}}\,\mathrm{d}\Omega\end{aligned} \tag{12}$$

for $q = 0, 1, \ldots, M$, $r = 0, 1, \ldots, N$. ∇J can be computed from Eqs. (12), since the sensitivity coefficient vector, the exit radiation intensities, and the measured data are available.

Stopping Criterion

If the problem contains no measurement errors, the following condition

$$J(\boldsymbol{b}^{k+1}) < \boldsymbol{\delta}^* \tag{13}$$

can be used for terminating the iterative process, where $\boldsymbol{\delta}^*$ is a small specified positive number. However, the measured radiation intensities contain measurement errors. Following the computational experience, we use the discrepancy principle[21]

$$J(\boldsymbol{b}^{k+1}) < 8\pi\sigma^2 \tag{14}$$

as the stopping criterion, where σ is the standard deviation of the measurement errors.

Computational Algorithm

The computational procedure for the solution of the inverse problem can be summarized as follows:

Step 1: Pick an initial guess $\boldsymbol{b}^0$. Set $k = 0$.

Step 2: Solve the sensitivity problem given by Eqs. (11), and compute ∇I.

Step 3: Solve the direct problem given by Eqs. (1), and compute the exit radiation intensities $I_1(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_2(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_3(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, and $I_4(\boldsymbol{\Omega}; \boldsymbol{b}^k)$.

Step 4: Calculate the objective function given by Eq. (4). Terminate the iteration process if the specified stopping criterion is satisfied. Otherwise, go to step 5.

Step 5: Knowing ∇I, $I_1(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_2(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_3(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_4(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $Y_1(\boldsymbol{\Omega})$, $Y_2(\boldsymbol{\Omega})$, $Y_3(\boldsymbol{\Omega})$, and $Y_4(\boldsymbol{\Omega})$, compute the gradient $\nabla J(\boldsymbol{b}^k)$ from Eqs. (12).

Step 6: Knowing $\nabla J(\boldsymbol{b}^k)$, compute γ^k from Eq. (7), then compute $\boldsymbol{d}^k$ from Eq. (6).

Step 7: Knowing ∇I, $I_1(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_2(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_3(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $I_4(\boldsymbol{\Omega}; \boldsymbol{b}^k)$, $Y_1(\boldsymbol{\Omega})$, $Y_2(\boldsymbol{\Omega})$, $Y_3(\boldsymbol{\Omega})$, $Y_4(\boldsymbol{\Omega})$, and $\boldsymbol{d}^k$, compute $\boldsymbol{\beta}^k$ from Eq. (9).

Step 8: Knowing $\boldsymbol{\beta}^k$ and $\boldsymbol{d}^k$, compute $\boldsymbol{b}^{k+1}$ from Eq. (5). Set $k = k + 1$ and go to step 3. To start the iteration, an initial guess $\boldsymbol{b}^0 = \boldsymbol{0}$ is used.

Results and Discussion

The inverse problem of concern here is to identify the unknown source term from the knowledge of the exit radiation intensities at the centers of the boundaries. To examine the effectiveness of the present method, several test cases are considered. For illustration, we use the polynomials $[y^q]$ and $[z^r]$ as the basis functions $[f_q(y)]$ and $[g_r(z)]$, respectively. The simulated measured exit radiation intensities Y_i ($i = 1, 2, 3, 4$) are generated by adding error terms to the exact radiation intensities

$$(Y_i)_{\text{measured}} = (Y_i)_{\text{exact}} + \sigma\zeta \qquad i = 1, 2, 3, 4 \tag{15}$$

where σ is the standard deviation of the measurement data, and ζ is a normal distributed random variable with zero mean and unit standard deviation. There is a 99% probability of ζ lying in the range $-2.576 \leq \zeta \leq 2.576$. For all of the results presented in this paper, we assume that the exit radiation intensities are available at the quadrature points for the S_6 method. The estimated and exact values of the source terms for the following test cases are close to one another in graphic forms, and so only the errors of the inverse solutions are presented.

We first consider a source term represented as

$$S_1(\boldsymbol{\tau}) = 5y^4z^4 + 2y^2 + 4z^2 + 3y + 6z + 1$$
$$0 \leq y \leq 1, \qquad 0 \leq z \leq 1 \tag{16}$$

and the medium is isotropic scattering with a single-scattering albedo 0.5. The exact function for the source term is shown in Fig. 3. Table 2 shows the rms errors of the estimated results for the source term by the inverse analysis, using measured data with $\sigma = 0$, 0.06, and 0.12. The rms error is defined as

$$\text{rms error} = \left\{\frac{1}{HW}\int_0^H\int_0^W [S_{\text{estimated}}(\boldsymbol{\tau}) - S_{\text{exact}}(\boldsymbol{\tau})]^2\,\mathrm{d}y\,\mathrm{d}z\right\}^{1/2} \tag{17}$$

It is noted from Table 2 that with no measurement errors, $\sigma = 0$, the agreement between the estimated and the exact values of the source term is excellent. The effects of the measurement errors on the estimation are also shown. The simulated experimental data containing errors of standard deviation

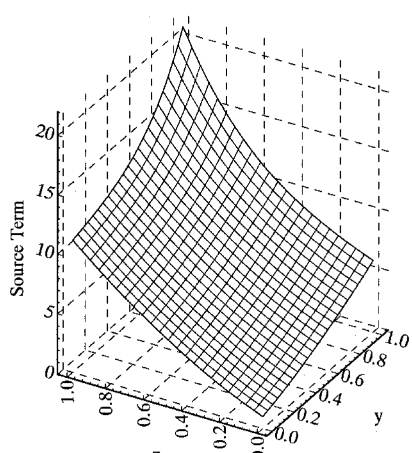

Fig. 3 Exact function for the source term $S_1(\boldsymbol{\tau})$.

Table 2 RMS error of the estimation for the source term $S_1(\tau)$, with isotropic scattering, $\omega = 0.5$, $W = H = 1$

σ	RMS
0	0.000
0.06	0.212
0.12	0.248

Table 3 Expansion coefficients for the scattering phase functions

a_n	Forward scattering	Backward scattering
0	1.0	1.0
1	1.98398	−1.2
2	1.50823	0.5
3	0.70075	——
4	0.23489	——
5	0.05133	——
6	0.00760	——
7	0.00048	——

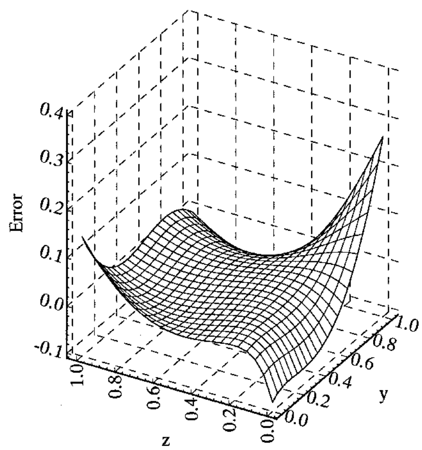

Fig. 4 Relative error of the estimation for the source term $S_1(\tau)$, with isotropic scattering, $\omega = 0.5$, $W = H = 1$, $\sigma = 0.12$.

$\sigma = 0.06$ and 0.12 correspond to maximum measurement errors of 4.8 and 9.6%, respectively. Increasing σ from 0.06 to 0.12, the accuracy of the estimation decreases. To show the accuracy of the estimation more clearly, the relative error defined as

$$\text{relative error} = \frac{S_{\text{estimated}}(\boldsymbol{\tau}) - S_{\text{exact}}(\boldsymbol{\tau})}{S_{\text{exact}}(\boldsymbol{\tau})} \tag{18}$$

is plotted in Fig. 4 for $\sigma = 0.12$. Because the measured exit radiation intensities are available only at the centers of the boundary surfaces, the errors of the estimation are larger near the corners, and the maximum error is 26.7%. Even in this case, the estimated and exact values of the source term are close to each other in graphic form.

To show the effects of anisotropic scattering on the source term estimation, two different scattering laws, one representing forward scattering and the other backward scattering, are considered as listed in Table 3. The rms errors of the inverse solutions are shown in Tables 4 and 5 for the forward- and backward-scattering cases, respectively. Compared with Table 2, we note that the effects of the anisotropic scattering on the accuracy of the estimation are not very significant, and the results obtained with the inverse analysis are good. If cases with severe scattering anisotropy are considered, a higher-order S_N method should be used to solve the direct problem.

Tables 6 and 7 are intended to show the effects of the single-scattering albedo on the accuracy of the estimation. The rms errors of the inverse solutions for isotropic scattering media with single-scattering albedos of 0.1 and 0.9 are shown in Tables 6 and 7, respectively. From Tables 6 and 7 we note that the reconstruction of the source term is good for exact and noisy input data.

We now consider the effects of the optical thickness on the source term estimation. In this case, the medium is assumed

Table 4 RMS error of the estimation for the source term $S_1(\tau)$, with forward scattering, $\omega = 0.5$, $W = H = 1$

σ	RMS
0	0.000
0.06	0.092
0.12	0.216

Table 5 RMS error of the estimation for the source term $S_1(\tau)$, with backward scattering, $\omega = 0.5$, $W = H = 1$

σ	RMS
0	0.000
0.06	0.160
0.12	0.233

Table 6 RMS error of the estimation for the source term $S_1(\tau)$, with isotropic scattering, $\omega = 0.1$, $W = H = 1$

σ	RMS
0	0.000
0.06	0.203
0.12	0.255

Table 7 RMS error of the estimation for the source term $S_1(\tau)$, with isotropic scattering, $\omega = 0.9$, $W = H = 1$

σ	RMS
0	0.000
0.06	0.137
0.12	0.178

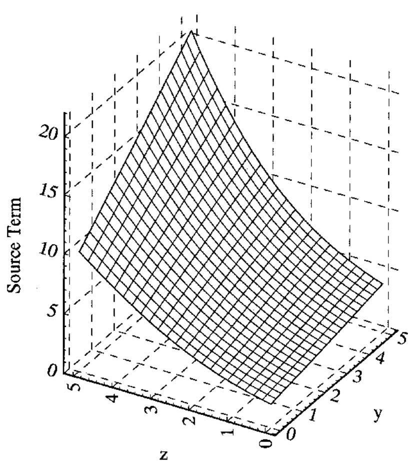

Fig. 5 Exact function for the source term $S_2(\tau)$.

Table 8 RMS error of the estimation for the source term $S_2(\boldsymbol{\tau})$, with isotropic scattering, $\omega = 0.5$, $W = H = 5$

σ	RMS
0	0.000
0.2	0.077
0.6	0.231

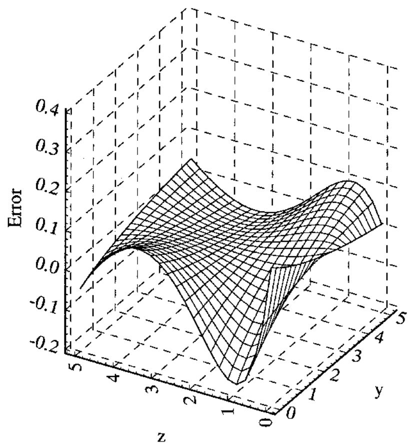

Fig. 6 Relative error of the estimation for the source term $S_2(\boldsymbol{\tau})$, with isotropic scattering, $\omega = 0.5$, $W = H = 5$, $\sigma = 0.6$.

to be isotropic scattering with a single-scattering albedo 0.5, and the unknown source term is expressed as

$$S_2(\boldsymbol{\tau}) = 0.01yz^3 + 0.2z^2 + 0.1yz + 0.4y + 0.6z + 2$$
$$0 \le y \le 5, \qquad 0 \le z \le 5 \qquad (19)$$

The exact function for the source term is plotted in Fig 5. Both exact and noisy input data are used to illustrate the performance of the inverse method as shown in Table 8. The use of exact input data, $\sigma = 0$, for the inverse analysis, produces excellent agreement between the exact and estimated source term. In the case of input data containing errors of standard deviation $\sigma = 0.2$ and 0.6, the results obtained with the method are good, even in the presence of noisy data. $\sigma = 0.2$ and 0.6 correspond to maximum errors of 6.6 and 19.8% in the measured exit radiation intensity data, respectively. Figure 6 shows the relative error of the estimated source term for $\sigma = 0.6$. The maximum error is -17.9% near the corner. The CPU time required for each case varied from 50 s to 33 min on a personal computer with an Intel Pentium Pro 200 MHz processor.

Conclusions

The conjugate gradient method has been applied to solve the inverse radiation problem for estimating the source term in a two-dimensional absorbing, emitting, scattering rectangular medium from the knowledge of the radiation intensities exiting the boundaries. Although the source term is a function of the space variables, the inverse algorithm presented requires measurements of the outgoing radiation intensities only at the center of each boundary. Both exact and noisy input data have been used to test the performance of the proposed method. The effects of the anisotropic scattering, single-scattering albedo, and optical thickness on the accuracy of the estimation have been considered. The results of this study show that the source term can be estimated accurately, even with noisy data. The algorithm can be extended to three-dimensional geometry with partially reflecting surfaces. However, the computational time may increase significantly.

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